Section 15.5

Applications of Multiple Integrals

Density and Mass

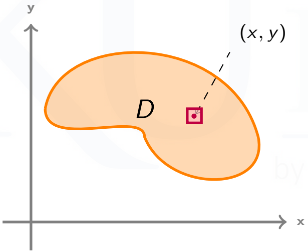
Moments and Centers of Mass

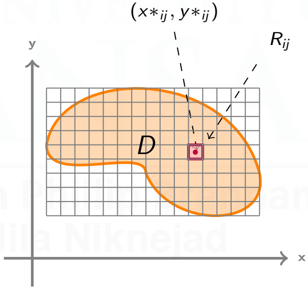
- Example, 3-dimensional

1 Density and Mass

Density and Mass

Suppose a lamina occupies a region D of the xy-plane and its density (in units of mass per unit area) at a point (x, y) in D is given by $\delta(x, y)$, where δ is a continuous function on D.





$$\text{Mass} \; = \lim_{(n,m)\to(\infty,\infty)} \sum_{i=1}^{n} \sum_{j=1}^{m} \delta(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_D \delta(x, y)\, dA$$

Density and Mass

Similarly, if a solid body occupies a region $S \subset \mathbb{R}^3$ and its density (in mass per unit volume) at a point (x, y, z) in S is $\delta(x, y, z)$, then its total mass is

$$\iiint_S \delta(x, y, z)\, dV.$$

Note: Some sources use σ for density in the plane and ρ for density in 3-space.

Example 1: Find the mass of the rectangle $R = [0, 2] \times [0, 3]$ with density $\delta(x, y) = xy^2$ kg/m^2.

Solution:

$$\iint_R \delta(x, y)\, dA = \int_0^3 \int_0^2 xy^2\, dx\, dy$$

$$= \left(\int_0^2 x\, dx\right) \left(\int_0^3 y^2\, dy\right) = 2 \cdot 3 = 6 \text{ kg}$$

2 Moments and Centers of Mass

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Moments and Center of Mass

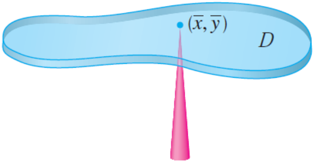
The **moments** M_x and M_y of a lamina measure how balanced it is with respect to the x- and y-axes.

$$M_x = \iint_D y\delta(x,y)\,dA \qquad M_y = \iint_D x\delta(x,y)\,dA$$

where D is the region occupied by the lamina.
The coordinates $(\overline{x}, \overline{y})$ of the center of mass are

$$\overline{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\delta(x,y)\,dA$$

$$\overline{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\delta(x,y)\,dA$$



Think of $\overline{x}$ and $\overline{y}$ as weighted averages: the factor δ assigns more weight to points with larger mass density.

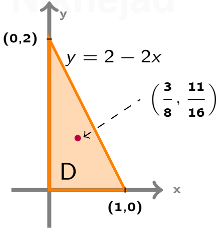
Example 2: Find the mass and center of mass of a triangular lamina with vertices $(0,0)$, $(1,0)$, and $(0,2)$ and density function $\delta(x,y) = 1 + 3x + y$. Solution: The lamina is bounded by $y = 2 - 2x$, $y = 0$, and $x = 0$.
Its mass is

$$m = \int\int_D \delta(x,y)\,dA = \int_0^1 \int_0^{2-2x} (1 + 3x + y)\,dy\,dx = \frac{8}{3}$$

Center of mass:

$$\begin{aligned}
\overline{x} &= \frac{1}{m}\int_0^1 \int_0^{2-2x} x(1+3x+y)\,dy\,dx \\
&= \frac{1}{m}\int_0^1 \int_0^{2-2x} (x + 3x^2 + xy)\,dy\,dx \\
&= \frac{1}{m}\int_0^1 \left(xy + 3x^2y + xy^2/2\right)\Big|_0^{2-2x}\,dx \\
&= \frac{1}{m}\int_0^1 \left(x(2-2x) + 3x^2(2-2x) + x(2-2x)^2/2\right)\,dx \\
&= \boxed{3/8}
\end{aligned}$$

$$\begin{aligned}
\overline{y} &= \frac{1}{m}\int_0^1 \int_0^{2-2x} y(1+3x+y)\,dy\,dx \\
&= \boxed{11/16}
\end{aligned}$$



Moments and Center of Mass

In $\mathbb{R}^3$, the moments of a solid S are defined not with respect to the axes as in $\mathbb{R}^2$, but with respect to the coordinate planes:

$$M_{yz} = \iiint_S x\delta(x,y,z)\,dV \qquad \overline{x} = \frac{M_{yz}}{m}$$

$$M_{xz} = \iiint_S y\delta(x,y,z)\,dV \qquad \overline{y} = \frac{M_{xz}}{m}$$

$$M_{xy} = \iiint_S z\delta(x,y,z)\,dV \qquad \overline{z} = \frac{M_{xy}}{m}$$

Again, the moments measure how balanced the solid is with respect to each of the coordinate planes.

Example 3: Find the center of mass of the tetrahedron S of uniform density bounded by the coordinate planes and the plane $x + y + z = 1$.

<u>Solution:</u> Let δ be the density of S (so δ is a constant). The mass of S is

$$\iiint_S \delta \, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \delta \, dz \, dy \, dx = \frac{\delta}{6}$$

$$M_{yz} = \iiint_S x\delta \, dV = \frac{\delta}{24} \qquad \overline{x} = \frac{1}{4}$$

$$M_{xz} = \iiint_S y\delta \, dV = \frac{\delta}{24} \qquad \overline{y} = \frac{1}{4}$$

$$M_{xy} = \iiint_S z\delta \, dV = \frac{\delta}{24} \qquad \overline{z} = \frac{1}{4}$$